DEFORMATION MAP FOR GENERALIZED κ-POINCARÉ AND κ-WEYL ALGEBRAS

Stefan Giller*, Cezary Gonera*, Michał Majewski*
Department of Theoretical Physics
University of Łódź
ul. Pomorska 149/153, 90–236 Łódź, Poland

Abstract. A nonlinear transformation in the momentum space is constructed which converts the deformed action of Lorentz and Weyl generators on momenta into the standard one.

1. Introduction

The κ-Poincaré algebra was introduced by Lukierski, Nowicki and Ruegg [1], [2]. This Hopf algebra deformation of classical Poincaré algebra seems to be quite interesting since it depends on dimensionful parameter κ. Many formal properties of κ-Poincaré algebra were analysed in a number of recent paper. In particular, Majid and Ruegg [3] have shown that it possesses a biocrossproduct structure, one of the factors being the classical Lorentz algebra while remaining one — a deformed (in coalgebra sector) momentum algebra. In the algebraic sector the whole deformation consists in nonlinear action of Lorentz generators on momenta. It appears ([2], [4]) that one can find a nonlinear change of variables in momentum space which reduces the 'deformed' action of Lorentz generators to the standard one. The existence of such a mapping ('deformation map') does not mean that the deformed algebra is equivalnet to the standard one — the coalgebraic sectors are not mapped onto each other.

Quite recently([5]), the κ-deformed Poincaré algebra has been generalized to the case of inhomogeneous algebra acting in arbitrary n-dimensional flat space. We do not assume any longer that $g_{\mu\nu} = \mathrm{diag}(+, -, \dots, -)$; on the contrary, $g_{\mu\nu}$ is now arbitrary symmetry invertible matrix.

* Supported by KBN grant 2 P 3022 1706 p 02

The resulting algebra reads

$$
\begin{aligned}
&[M^{\mu\nu}, M^{\alpha\beta}] = i(g^{\mu\beta}M^{\nu\alpha} - g^{\nu\beta}M^{\mu\alpha} + g^{\nu\alpha}M^{\mu\beta} - g^{\mu\alpha}M^{\nu\beta}),\\
&[\widetilde{P}_\mu, \widetilde{P}_\nu] = 0,\\
&[M^{ij}, \widetilde{P}_0] = 0,\\
&[M^{ij}, \widetilde{P}_k] = i\kappa({\delta^j}_k g^{0i} - {\delta^i}_k g^{0j})(1 - e^{-\widetilde{P}_0/\kappa}) + i({\delta^j}_k g^{is} - {\delta^i}_k g^{js})\widetilde{P}_s,\\
&[M^{i0}, \widetilde{P}_0] = i\kappa g^{i0}(1 - e^{-\widetilde{P}_0/\kappa}) + ig^{ik}\widetilde{P}_k,\\
&[M^{i0}, \widetilde{P}_k] = -\frac{i\kappa}{2}g^{00}{\delta^i}_k(1 - e^{-2\widetilde{P}_0/\kappa}) - i{\delta^i}_k g^{0s}\widetilde{P}_s e^{-\widetilde{P}_0/\kappa}\\
&\qquad + ig^{0i}\widetilde{P}_k(e^{-\widetilde{P}_0/\kappa} - 1) + \frac{i}{2\kappa}{\delta^i}_k g^{rs}\widetilde{P}_r\widetilde{P}_s - \frac{i}{\kappa}g^{is}\widetilde{P}_s\widetilde{P}_k
\end{aligned}
\tag{1a}
$$

and

$$
\begin{aligned}
&\Delta\widetilde{P}_0 = I \otimes \widetilde{P}_0 + \widetilde{P}_0 \otimes I,\\
&\Delta\widetilde{P}_k = \widetilde{P}_k \otimes e^{-\widetilde{P}_0/\kappa} + I \otimes \widetilde{P}_k,\\
&\Delta M^{ij} = M^{ij} \otimes I + I \otimes M^{ij},\\
&\Delta M^{i0} = I \otimes M^{i0} + M^{i0} \otimes e^{-\widetilde{P}_0/\kappa} - \frac{1}{\kappa}M^{ij} \otimes \widetilde{P}_j
\end{aligned}
\tag{1b}
$$

with $i, j, k, \ldots = 1, 2, \ldots, n-1$.

It was also shown ([5]) that, provided $g_{00} = 0$ (which excludes the positive-definite metric $g_{\mu\nu}$), algebra (1) can be extended to κ-deformed Weyl algebra. The additional operator D (dilatation) commutus with $M_{\mu\nu}$ and obeys

$$
\begin{aligned}
&[D, \widetilde{P}_0] = i\kappa(1 - e^{-\widetilde{P}_0/\kappa}),\\
&[D, \widetilde{P}_i] = i\widetilde{P}_i e^{-\widetilde{P}_0/\kappa} + ig_{0i}g^{0s}\widetilde{P}_s(1 - e^{-\widetilde{P}_0/\kappa}) + \frac{i}{2\kappa}g_{0i}g^{rs}\widetilde{P}_r\widetilde{P}_s\\
&\qquad + \frac{i\kappa}{2}g^{00}g_{i0}(1 - e^{-\widetilde{P}_0/\kappa})^2,\\
&\Delta D = D \otimes I + I \otimes D + g_{0i}M^{i0} \otimes (1 - e^{-\widetilde{P}_0/\kappa}) - \frac{1}{\kappa}g_{0i}M^{ik} \otimes \widetilde{P}_k.
\end{aligned}
\tag{2}
$$

In the present paper we extend the construction of deformation map given in [4] to cover algebras (1) and (2). Such a deformation map is very useful in classifying the representations of deformed algebras. Apart from this it provides an independent consistency check (of algebraic sector of) of algebras (1) and (2).

II. Deformation map

In this section we will generalize to the case of general κ-Poincaré algebra as well as κ-Weyl algebra the deformation map obtained in [4]. To this end let P_μ and $M^{\mu\nu}$ be the generators of classical Poincaré algebra. We will be looking for the deformation map of the following form

$$
\begin{aligned}
&\widetilde{P}_0 = g(P_0, M^2),\\
&\widetilde{P}_k = f(P_0, M^2)P_k + g_{k0}h(P_0, M^2);
\end{aligned}
\tag{3}
$$

here M^2 is the standard mass squared Casimir, $M^2 = g^{\mu\nu}P_\mu P_\nu$. Inserting (3) into (1a) we arrive at the following set of equations (primes denotes differentiation with respect to P_0)

$$
\begin{aligned}
&g' P_0 = \kappa(1 - e^{-g/\kappa}) - g_{00}h,\\
&g' = f,\\
&f P_0 = \kappa(1 - e^{-g/\kappa}) - g_{00}h,\\
&f' P_0 = \frac{1}{\kappa} f h g_{00} + (e^{-g/\kappa} - 1) f,\\
&f' = -\frac{1}{\kappa} f^2,\\
&\frac{1}{\kappa} f h = -h',\\
&f P_0 = \frac{\kappa}{2}(1 - e^{-2g/\kappa}) - g_{00} h e^{-g/\kappa} - \frac{1}{2\kappa} g_{00} h^2 + \frac{1}{2\kappa} P_0^2 f^2,\\
&f = e^{-g/\kappa} f + \frac{1}{\kappa} g_{00} f h + \frac{1}{\kappa} P_0 f^2,\\
&h' P_0 = h(e^{-g/\kappa} - 1) + \frac{1}{\kappa} g_{00} h^2,\\
&h e^{-g/\kappa} = \frac{1}{2\kappa} M^2 f^2 - \frac{1}{2\kappa} g_{00} h^2.
\end{aligned}
\tag{4}
$$

These equations can be solved to yield

$$
\begin{aligned}
f &= \frac{\kappa}{P_0 + C(M^2)},\\
h &= \frac{\kappa A(M^2)}{P_0 + C(M^2)},\\
g &= \kappa \ln\Big(\frac{P_0 + C(M^2)}{C - g_{00} A(M^2)}\Big)
\end{aligned}
\tag{5}
$$

with A and C subjected to the condition

$$
g_{00} A^2(M^2) - 2A(M^2) C(M^2) + M^2 = 0. \tag{6}
$$

Finally, the deformation map reads

$$
\begin{aligned}
\widetilde{P}_0 &= \kappa \ln\Big(\frac{P_0 + C}{C - g_{00} A}\Big),\\
\widetilde{P}_i &= \frac{\kappa P_i}{P_0 + C} + \frac{\kappa A}{P_0 + C} g_{i0}.
\end{aligned}
\tag{7}
$$

The inverse map takes the form

$$
\begin{aligned}
P_0 &= (C - g_{00} A) e^{\widetilde{P}_0/\kappa} - C,\\
P_i &= \frac{C - g_{00} A}{\kappa} e^{\widetilde{P}_0/\kappa} \widetilde{P}_i - g_{i0} A.
\end{aligned}
\tag{8}
$$

Let us insert these expressions into the formula for the Casimir operator M^2. After some algebra we obtain

$$\begin{aligned}\frac{2A}{C-g_{00}A} = \frac{1}{\kappa^2}\Big[g^{00}\Big(2\kappa\,\mathrm{sh}\Big(\frac{\widetilde{P}_0}{\kappa}\Big)\Big)^2 + 4\kappa g^{0l}\widetilde{P}_l e^{\widetilde{P}_0/2\kappa}\,\mathrm{sh}\Big(\frac{\widetilde{P}_0}{2\kappa}\Big) \\ + g^{rs}\widetilde{P}_r e^{\widetilde{P}_0/2\kappa}\widetilde{P}_s e^{\widetilde{P}_0/2\kappa}\Big].\end{aligned} \tag{9}$$

The left-hand side depends only on M^2 so we can define the deformed mass square Casimir as

$$\begin{aligned}\widetilde{M}^2 = g^{00}\Big(2\kappa\,\mathrm{sh}\Big(\frac{\widetilde{P}_0}{\kappa}\Big)\Big)^2 + 4\kappa g^{0l}\widetilde{P}_l e^{\widetilde{P}_0/2\kappa}\,\mathrm{sh}\Big(\frac{\widetilde{P}_0}{2\kappa}\Big) \\ + g^{rs}\widetilde{P}_r e^{\widetilde{P}_0/2\kappa}\widetilde{P}_s e^{\widetilde{P}_0/2\kappa}.\end{aligned} \tag{10}$$

Obviously, $\widetilde{M}^2 \to M^2$ as $\kappa \to \infty$. he following relation between M^2 and $\widetilde{M}^2$ arises as a consequence of (6) and (9)

$$A^2\Big(\frac{4\kappa^2}{\widetilde{M}^2} + g_{00}\Big) = M^2. \tag{11}$$

In principle, we can also find the second Casimir operator — the Pauli-Lubanski invariant. However, the resulting formula is quite involved and will be not quoted here.

Now let us consider the case of Weyl algebra. As we stressed before, it is consistent only provided $g_{00} = 0$. Since the κ-Poincaré algebra is a subalgebra of κ-Weyl algebra, our formulae (7) are still valid. However, A and C should be subjected to some further conditions providing the proper action of dilatation operator D on new momenta. It appears, as a result of simple computation, that C should be M^2-independent constant. If we put $C = \kappa$, we obtain finally the following deformation map

$$\begin{aligned}\widetilde{P}_0 &= \kappa\ln\Big(\frac{P_0+\kappa}{\kappa}\Big), \\ \widetilde{P}_i &= \frac{\kappa}{P_0+\kappa}P_i + \frac{M^2}{2(P_0+\kappa)}g_{i0}.\end{aligned} \tag{13}$$

We have shown that formally the classical Poincaré (Weyl) algebra and the κ-Poincaré (κ-Weyl) algebra are equivalent in the algebraic sector. This is no longer true in the coalgebra sector: the standard Poincaré algebra is cocommutative while the κ-Poincaré one is not.

Acknowledgment.
The numerous discussions with P. Kosiński and P. Maślanka are kindly acknowledged.

References

[1] J. Lukierski, A. Novicki, H. Ruegg,, Phys. Lett. **B 293** (1992), 344.
[2] J. Lukierski, H. Ruegg, V. Tolstoy, in: Quantum Groups. Formalism and Applications. Proc. of XXX Karpacz Winter School of Theoretical Physics, ed. J. Lukierski, Z. Popowicz, J. Sobczyk, PWN 1995.
[3] S. Majid, H. Ruegg, Phys. Lett. **B 334** (1994), 348.
[4] J. Lukierski, P. Kosiński, P. Maślanka, J. Sobczyk, *The classical basis for κ-deformed Poincaré subalgebra and the second κ-deformed supersymmetric Casimir*, Mod. Phys. Lett. A 10 (1995), 2599.
[5] P. Kosiński, P. Maślanka, *The κ-Weyl group and its algebra*, preprint IMUŁ, Łódź University 1995, to be published.